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Two-dimensional algebro-geometric difference operators

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Abstract. A generalized inverse problem for a two-dimensional difference operator is introduced. A new construction of the algebro-geometric difference operators of two types first considered by Krichever and by Novikov is proposed.

1. Introduction

The notion of a finite-gap with respect to a one energy level Schrödinger operator was introduced by Dubrovin *et al* in [1]. Novikov and Veselov [2] dealt with a class of two-dimensional Schrödinger operators called potential operators and solved the inverse scattering problem. Krichever [3] introduced a similar theory for difference operators. Recent work [4–6] has dealt with different natural generalizations of two-dimensional difference operators defined on regular graphs and lattices. In particular, in [4] (see also appendix I in [5]) in the context of discrete Laplace transformations Novikov introduced an important class of difference operators on a equilateral triangular lattice. These papers stimulated new research in this area (see the review in [6]).

In this paper we propose a generalized inverse problem and a new construction of two-dimensional algebro-geometric operators in Krichever's and Novikov's classes of operators.

Let L be a two-dimensional difference operator (of order $2K$)

$$(L\psi)_{nm} = \sum_{i,j,|i|\leqslant K,|j|\leqslant K} a^{ij}_{nm}\psi_{n+i,m+j} \qquad (1)$$

with periodic coefficients

$$a^{ij}_{n+N,m} = a^{ij}_{n,m+M} = a^{ij}_{nm}.$$

Consider a space of Floquet functions

$$\psi_{n+N,m} = w_1\psi_{n,m} \qquad \psi_{n,m+M} = w_2\psi_{n,m}.$$

This space is finite-dimensional and the operator L induces in this space a linear operator $L(w_1, w_2)$. The characteristic equation of this operator

$$Q(w_1, w_2, E) = \det(E \cdot \mathrm{Id} - L(w_1, w_2)) = 0$$

defines a two-dimensional algebraic variety M^2. A point M^2 corresponds to a unique eigenvector ψ_{nm} of the operator L

$$(L\psi)_{nm} = E\psi_{nm}$$

such that $\psi_{00} = 1$. All other components ψ_{nm} are meromorphic functions on M^2. Consider a curve $\Gamma \subset M^2$ corresponding to the 'zero-energy level'

$$\Gamma = \{w_1, w_2 | Q(w_1, w_2, 0) = 0\}.$$

The functions ψ_{nm} are meromorphic on Γ.

We can consider the two following problems.

(1) The direct spectral problem. Find explicitly the 'spectral data' of the operator L (i.e. a set of geometric data like a curve Γ, divisors of poles of ψ_{nm} etc) which determines the operator L uniquely.
(2) The inverse spectral problem. Find explicitly the operator L using the 'spectral data'.

Both problems are complicated. It is nearly impossible to solve either of them in a general case. We can, however, consider a generalized inverse problem which consists of finding a set of geometric data with the following properties:

(1) The set of geometric data uniquely defines a family of functions ψ_{nm} defined on an algebraic complex curve Γ.
(2) These functions satisfy the equation $L\psi = 0$ for some operator L of the form (1).
(3) The operator L is uniquely defined by the equation $L\psi = 0$ and the coefficients a_{nm}^{ij} can be found explicitly.

This problem is solved for some particular operators in the paper [3]. Krichever calls such operators 'integrable' but we will use the term 'algebro-geometric'.

Our goal is to find algebro-geometric operators. We found two examples which can be of interest.

The first example is provided by operators of the form

$$(L\psi)_{nm} = a_{nm}\psi_{n-1,m} + b_{nm}\psi_{n+1,m} + c_{nm}\psi_{n,m-1} + d_{nm}\psi_{n,m+1} + v_{nm}\psi_{nm}. \quad (2)$$

A value of $(L\psi)_{nm}$ depends only on values of ψ at the points

$$(n-1, m) \qquad (n+1, m) \qquad (n, m-1) \qquad (n, m+1) \qquad (n, m)$$

which form a cross in the plane (n, m). We will call such an operator 'cross-shaped'. These operators were considered by Krichever in [3]. Algebro-geometric operators of the form (2) found by Krichever correspond to a curve $\Gamma' \subset M^2$ whose image under projection on the E-plane is the whole E-plane. The corresponding problem is $L\psi = E\psi$, where both E and ψ are functions defined on Γ'. In this paper we deal with a different type of algebro-geometric operators of the form (2) which corresponds to the 'zero-energy level' curve. The corresponding problem is $L\psi = 0$.

The other example is more complicated and perhaps more interesting. Consider a triangular lattice in a plane. We will use as coordinates triples of integers k, l, m such that $k + l + m = 0$. On such a lattice we can consider an operator of the form

$$\begin{aligned}(L\psi)_{klm} = {} & a_{klm}\psi_{k,l+1,m-1} + b_{klm}\psi_{k,l-1,m+1} + c_{klm}\psi_{k+1,l-1,m} \\ & + d_{klm}\psi_{k-1,l+1,m} + f_{klm}\psi_{k+1,l,m-1} + g_{klm}\psi_{k-1,l,m+1}. \end{aligned} \quad (3)$$

A value of $(L\psi)_{klm}$ depends only on values of ψ at the points

$$\begin{array}{lll}(k, l+1, m-1) & (k, l-1, m+1) & (k+1, l-1, m) \\ (k-1, l+1, m) & (k+1, l, m-1) & (k-1, l, m+1)\end{array}$$

which form a hexagon in the plane k, l, m. We will call such an operator 'hexagonal'. In this case our lattice is not rectangular, nevertheless we can consider the generalized inverse problem and solve it.

As we have already mentioned, this class of operators has been introduced by Novikov [4–6] in the context of the discrete Laplace transformation.

It should be noted that our formulae in sections 3 and 4 are not unique. We can choose other singularity structures for the ψ-function (for example rotating the plane (n, m) by $\frac{\pi}{2}$ in the case of operators of the form (2)) and obtain other algebro-geometric operators.

2. Notation and conventions

We use the notations and conventions of [7]. In particular, our conventions are as follows. A basis of cycles $a_1, \ldots, a_g, b_1, \ldots, b_g$ is chosen in such a way that

$$a_i \circ a_j = b_i \circ b_j = 0 \qquad a_i \circ b_j = \delta_{ij} \qquad i, j = 1, \ldots, g$$

where g is the genus of a non-singular curve Γ. A basis of holomorphic differentials $\omega_1, \ldots, \omega_g$ is chosen in such a way that

$$\oint_{a_j} \omega_k = 2\pi \mathrm{i} \delta_{jk} \qquad j, k = 1, \ldots, g.$$

We define the Jacobian $J(\Gamma)$ as $\mathbb{C}^g/\{2\pi \mathrm{i} M + BN\}$, where $M, N \in \mathbb{Z}^g$, B is a matrix of b-periods of ω_i

$$B_{jk} = \oint_{b_j} \omega_k \qquad j, k = 1, \ldots, g.$$

We denote by Ω_{PQ} the Abel differential of the third kind, i.e. a differential with unique poles at the points P and Q and residues +1 and -1 at these points respectively, we denote by U_{PQ} the vector of the b-periods of Ω_{PQ} and by $\mathcal{K}$ the vector of the Riemann constants.

We define the Θ-function as

$$\Theta(z) = \sum_{N \in \mathbb{Z}^g} \exp\left(\tfrac{1}{2}\langle BN, N\rangle + \langle N, z\rangle\right)$$

where $z = (z_1, \ldots, z_g) \in \mathbb{C}^g$ and $\langle\, ,\, \rangle$ is a Euclidean scalar product $\langle x, y\rangle = \sum_{i=1}^{g} x_i y_j$.

We use the following natural convention: if n is a negative integer then a zero (pole) of the nth order is a pole (zero) of the $|n|$th order.

3. The cross-shaped operators: Krichever's class

Consider an arbitrary two-dimensional difference operator L of the form (2). Our goal is to find a solution of the generalized inverse problem stated in the introduction.

Our construction is as follows. Let Γ be a non-singular curve of genus g. Let $P_i^{\pm}$, $i = 1, 2, 3$, be six points on Γ. Let $\mathcal{D}$ be a generic divisor of the form $\mathcal{D} = P_1 + \cdots + P_g$ such that the points P_k are different from the $P_i^{\pm}$. Consider a function $\phi_{\alpha\beta\gamma}, \alpha, \beta, \gamma \in \mathbb{Z}$, defined on Γ such that:

(1) If a point $P \in \Gamma \setminus \{P_1^{\pm}, P_2^{\pm}, P_3^{\pm}\}$ is a pole of $\phi_{\alpha\beta\gamma}$, then P is one of the points P_k;

(2) The function $\phi_{\alpha\beta\gamma}$ has a zero of αth order in P_1^{+} and a pole of αth order in P_1^{-}, the same structure for β and $P_2^{\pm}$, γ and $P_3^{\pm}$.

Lemma.

(1) Such a function $\phi_{\alpha\beta\gamma}$ exists and is unique up to multiplication by a constant.

(2) The explicit formula for $\phi_{\alpha\beta\gamma}$ *is*

$$r_{\alpha\beta\gamma}\cdot\exp\int_{P_0}^{P}(\alpha\Omega_1+\beta\Omega_2+\gamma\Omega_3)\cdot\frac{\Theta(A(P)+\alpha U_1+\beta U_2+\gamma U_3-A(\mathcal{D})-\mathcal{K})}{\Theta(A(P)-A(\mathcal{D})-\mathcal{K})}$$

where $r_{\alpha\beta\gamma}$ *is an arbitrary constant,* P_0 *is a fixed point defining the Abel transform A (it should be remarked that the paths of integration in* $\int_{P_0}^{P}$ *and in the Abel transform are the same),* $\Omega_i=\Omega_{P_i^+P_i^-}$, $U_i=U_{P_i^+P_i^-}$.

Proof. The proof is performed by standard reasoning of the theory of the algebro-geometric integration. □

The key idea behind the construction of our functions ψ_{nm} is a convenient relabelling with $\psi_{nm}=\phi_{\alpha\beta\gamma}$, where

$$\left.\begin{aligned}\alpha(n,m)&=\frac{2-n-m}{2}\qquad \beta(n,m)=\frac{n-m}{2}\\ \gamma(n,m)&=\frac{n-m}{2}\end{aligned}\right\}\quad\text{if}\quad n+m=0\pmod 2$$

$$\left.\begin{aligned}\alpha(n,m)&=\frac{3-n-m}{2}\qquad \beta(n,m)=\frac{-1+n-m}{2}\\ \gamma(n,m)&=\frac{1+n-m}{2}\end{aligned}\right\}\quad\text{if}\quad n+m=1\pmod 2.$$

We will use vectorial notation for the triples, i.e. the representation of a triple α,β,γ as a vector $\alpha\boldsymbol{i}+\beta\boldsymbol{j}+\gamma\boldsymbol{k}$. For example, we will sometimes write $\phi_{\alpha\boldsymbol{i}+\beta\boldsymbol{j}+\gamma\boldsymbol{k}}$ instead of $\phi_{\alpha,\beta,\gamma}$. This is useful because, for example, if $\boldsymbol{v}=\alpha\boldsymbol{i}+\beta\boldsymbol{j}+\gamma\boldsymbol{k}$, then we can write $\phi_{\boldsymbol{v}+\boldsymbol{i}}$ instead of $\phi_{\alpha+1,\beta,\gamma}$.

We write $\psi_{nm}=\phi_{\boldsymbol{v}(n,m)}$, where $\boldsymbol{v}(n,m)=\alpha(n,m)\boldsymbol{i}+\beta(n,m)\boldsymbol{j}+\gamma(n,m)\boldsymbol{k}$, i.e.

$$\boldsymbol{v}(n,m)=\frac{2-n-m}{2}\boldsymbol{i}+\frac{n-m}{2}\boldsymbol{j}+\frac{n-m}{2}\boldsymbol{k}\qquad\text{if}\quad n+m=0\pmod 2$$

$$\boldsymbol{v}(n,m)=\frac{3-n-m}{2}\boldsymbol{i}+\frac{-1+n-m}{2}\boldsymbol{j}+\frac{1+n-m}{2}\boldsymbol{k}\qquad\text{if}\quad n+m=1\pmod 2.$$

We will also use the following notation:

$$\Theta(P,\alpha\boldsymbol{i}+\beta\boldsymbol{j}+\gamma\boldsymbol{k})=\Theta(P,\alpha,\beta,\gamma)=\Theta(A(P)+\alpha U_1+\beta U_2+\gamma U_3-A(\mathcal{D})-\mathcal{K}).$$

Let us formulate our theorem.

Theorem 1. *Let a family* ψ_{mn} *be defined as stated above. Then* $L\psi=0$ *if and only if the coefficients* a_{nm}, b_{nm}, c_{nm}, d_{nm}, v_{nm} *of the operator L are defined up to multiplication by a constant by the following formulae:*

(1) if $n+m\equiv 0\pmod 2$*, then*

$$\frac{a_{nm}}{d_{nm}}=-\frac{r_{\boldsymbol{v}-\boldsymbol{j}}}{r_{\boldsymbol{v}+\boldsymbol{i}-\boldsymbol{j}}}\cdot\frac{\Theta(P_2^+,\boldsymbol{v}-\boldsymbol{j})}{\Theta(P_2^+,\boldsymbol{v}+\boldsymbol{i}-\boldsymbol{j})}\cdot\exp\left(-\int_{P_0}^{P_2^+}\Omega_1\right)$$

$$\frac{b_{nm}}{d_{nm}}=-\frac{r_{\boldsymbol{v}-\boldsymbol{j}}}{r_{\boldsymbol{v}+\boldsymbol{k}}}\cdot\frac{\Theta(P_2^+,\boldsymbol{v}-\boldsymbol{j})\Theta(P_1^-,\boldsymbol{v}+\boldsymbol{i}-\boldsymbol{j})\Theta(P_3^-,\boldsymbol{v}+\boldsymbol{i}+\boldsymbol{k})}{\Theta(P_2^+,\boldsymbol{v}+\boldsymbol{i}-\boldsymbol{j})\Theta(P_1^-,\boldsymbol{v}+\boldsymbol{i}+\boldsymbol{k})\Theta(P_3^-,\boldsymbol{v}+\boldsymbol{k})}\times\exp\left(\int_{P_0}^{P_3^-}\Omega_1-\int_{P_0}^{P_2^+}\Omega_1-\int_{P_0}^{P_1^-}(\Omega_2+\Omega_3)\right)$$

$$\frac{c_{nm}}{d_{nm}}=\frac{r_{\boldsymbol{v}-\boldsymbol{j}}}{r_{\boldsymbol{v}+\boldsymbol{i}+\boldsymbol{k}}}\cdot\frac{\Theta(P_2^+,\boldsymbol{v}-\boldsymbol{j})\Theta(P_1^-,\boldsymbol{v}+\boldsymbol{i}-\boldsymbol{j})}{\Theta(P_2^+,\boldsymbol{v}+\boldsymbol{i}-\boldsymbol{j})\Theta(P_1^-,\boldsymbol{v}+\boldsymbol{i}+\boldsymbol{k})}\cdot\exp\left(-\int_{P_0}^{P_2^+}\Omega_1-\int_{P_0}^{P_1^-}(\Omega_2+\Omega_3)\right)$$

$$\frac{v_{nm}}{d_{nm}} = \frac{r_{v-j}}{r_v} \cdot \frac{\Theta(P_2^+, v-j)\Theta(P_1^-, v+i-j)}{\Theta(P_2^+, v+i-j)\Theta(P_1^-, v+i+k)}$$
$$\times \exp\left(-\int_{P_0}^{P_2^+}\Omega_1 - \int_{P_0}^{P_1^-}(\Omega_2+\Omega_3) + \int_{P_0}^{P_2^-}\Omega_3\right)$$
$$\times\left[\frac{\Theta(P_3^-, v+i+k)\Theta(P_2^-, v+k)}{\Theta(P_3^-, v+k)\Theta(P_2^-, v)} \cdot \exp\left(\int_{P_0}^{P_3^-}\Omega_1\right) - \frac{\Theta(P_2^-, v+i+k)}{\Theta(P_2^-, v)} \cdot \exp\left(\int_{P_0}^{P_2^-}\Omega_1\right)\right]$$

where $v = v(n, m)$*,*

(2) if $n + m \equiv 1 \pmod 2$*, then*

$$\frac{a_{nm}}{c_{nm}} = -\frac{r_{v+j}}{r_{v-k}} \cdot \frac{\Theta(P_2^-, v+j)\Theta(P_1^+, v-i+j)\Theta(P_3^+, v-i-k)}{\Theta(P_2^-, v-i+j)\Theta(P_1^+, v-i-k)\Theta(P_3^+, v-k)}$$
$$\times \exp\left(\int_{P_0}^{P_2^-}\Omega_1 + \int_{P_0}^{P_1^+}(\Omega_2+\Omega_3) - \int_{P_0}^{P_3^+}\Omega_1\right)$$

$$\frac{b_{nm}}{c_{nm}} = -\frac{r_{v+j}}{r_{v-i+j}} \cdot \frac{\Theta(P_2^-, v+j)}{\Theta(P_2^-, v-i+j)} \cdot \exp\left(\int_{P_0}^{P_2^-}\Omega_1\right)$$

$$\frac{d_{nm}}{c_{nm}} = \frac{r_{v+j}}{r_{v-i-k}} \cdot \frac{\Theta(P_2^-, v+j)\Theta(P_1^+, v-i+j)}{\Theta(P_2^-, v-i+j)\Theta(P_1^+, v-i-k)} \cdot \exp\left(\int_{P_0}^{P_2^-}\Omega_1 + \int_{P_0}^{P_1^+}(\Omega_2+\Omega_3)\right)$$

$$\frac{v_{nm}}{c_{nm}} = \frac{r_{v+j}}{r_v} \cdot \frac{\Theta(P_2^-, v+j)\Theta(P_1^+, v-i+j)}{\Theta(P_2^-, v-i+j)\Theta(P_1^+, v-i-k)}$$
$$\times \exp\left(\int_{P_0}^{P_2^-}\Omega_1 + \int_{P_0}^{P_1^+}(\Omega_2+\Omega_3) - \int_{P_0}^{P_{2+}}\Omega_3\right)$$
$$\times\left[\frac{\Theta(P_3^+, v-i-k)\Theta(P_2^+, v-k)}{\Theta(P_3^+, v-k)\Theta(P_2^+, v)} \cdot \exp\left(-\int_{P_0}^{P_3^+}\Omega_1\right) - \frac{\Theta(P_2^+, v-i-k)}{\Theta(P_2^+, v)} \cdot \exp\left(-\int_{P_0}^{P_2^+}\Omega_1\right)\right]$$

where $v = v(n, m)$*.*

Proof. Let $L\psi = 0$. Let us consider the case $n + m \equiv 0 \pmod 2$. Thus, the formula $L\psi = 0$ becomes

$$a_{nm}\phi_{v+i-j} + b_{nm}\phi_{v+k} + c_{nm}\phi_{v+i+k} + d_{nm}\phi_{v-j} + v_{nm}\phi_v = 0. \tag{4}$$

Consider the point P_1^-. Let λ be a local parameter in a neighbourhood of P_1^-. Hence $\phi_{\alpha\beta\gamma} = \lambda^{-\alpha} \cdot h$, where h is a holomorphic function. The function $\exp\int_{P_0}^{P}\Omega_1$ has a pole of first order at the point P_1^-. Thus $\exp\int_{P_0}^{P}\Omega_1 = K_1^-\lambda^{-1} + \cdots$, where K_1^- is a constant. Therefore we have

$$\phi_{\alpha\beta\gamma}(P) = r_{\alpha\beta\gamma}(K_1^-)^{\alpha}\left(\exp\int_{P_0}^{P_1^-}\Omega_2\right)^{\beta}\left(\exp\int_{P_0}^{P_1^-}\Omega_3\right)^{\gamma}\frac{\Theta(P_1^-, \alpha, \beta, \gamma)}{\Theta(P_1^-, 0, 0, 0)}\lambda^{-\alpha} + \cdots$$

for P in the neighbourhood of P_1^-.

Now we can write down the term with $\lambda^{-(\alpha(n,m)+1)}$ in formula (4) in the neighbourhood of P_1^-:

$$a_{nm}(K_1^-)^{\alpha(n,m)+1} r_{v+i-j}\left(\exp\int_{P_0}^{P_1^-}\Omega_2\right)^{\beta(n,m)-1}\left(\exp\int_{P_0}^{P_1^-}\Omega_3\right)^{\gamma(n,m)} \times\frac{\Theta(P_1^-,\boldsymbol{v}+\boldsymbol{i}-\boldsymbol{j})}{\Theta(P_1^-,0,0,0)}\lambda^{-(\alpha(n,m)+1)} + c_{nm}(K_1^-)^{\alpha(n,m)+1} r_{v+i+k}\left(\exp\int_{P_0}^{P_1^-}\Omega_2\right)^{\beta(n,m)}\left(\exp\int_{P_0}^{P_1^-}\Omega_3\right)^{\gamma(n,m)+1} \times\frac{\Theta(P_1^-,\boldsymbol{v}+\boldsymbol{i}+\boldsymbol{k})}{\Theta(P_1^-,0,0,0)}\lambda^{-(\alpha(n,m)+1)} = 0.$$

After simplification we obtain a linear equation for a_{nm} and c_{nm}:

$$a_{nm} r_{v+i-j}\Theta(P_1^-,\boldsymbol{v}+\boldsymbol{i}-\boldsymbol{j}) + c_{nm} r_{v+i+k}\exp\left(\int_{P_0}^{P_1^-}(\Omega_2+\Omega_3)\right)\Theta(P_1^-,\boldsymbol{v}+\boldsymbol{i}+\boldsymbol{k}) = 0.$$

By carrying out analogous computations at the points P_3^-, P_2^- and P_2^+ we obtain three other linear equations for a_{nm}, b_{nm}, c_{nm}, d_{nm} and v_{nm}. These equations can be explicitly solved and the formulae for the coefficients of the operator L given in the statement of the theorem are obtained. The case $n+m\equiv 1 \pmod 2$ is analogous.

Now let us suppose that $a_{nm},\dots,v_{nm}$ are given by the formulae in the theorem statement. Let us prove that $L\psi=0$. Consider the case $n+m\equiv 0 \pmod 2$. Let us consider a function

$$(\hat{L}\psi)_{nm} = \frac{a_{nm}}{d_{nm}}\psi_{n-1,m} + \frac{b_{nm}}{d_{nm}}\psi_{n+1,m} + \frac{c_{nm}}{d_{nm}}\psi_{n,m-1} + \frac{v_{nm}}{d_{nm}}\psi_{n,m}.$$

This function has the same pole or zero structure as the function $-\psi_{n,m+1}$ at the points P_i^+, $i=1,2,3$. It follows from the formulae in the theorem statement that $(\hat{L}\psi)_{nm}$ and $-\psi_{n,m+1}$ have the same pole or zero structure at P_i^-, $i=1,2,3$. If a point $P\in\Gamma\setminus\{P_1^\pm,P_2^\pm,P_3^\pm\}$ is a pole of $(\hat{L}\psi)_{nm}$ or $-\psi_{n,m+1}$, then $P\in\mathcal{D}$. Thus, by the lemma, $(\hat{L}\psi)_{nm}$ and $-\psi_{n,m+1}$ are proportional. Moreover, from the formulae for the coefficients of the operator L it follows that the terms with $\lambda^{\beta(n,m)-1}$ in the series expansions of these two functions at the point P_2^+ are the same. Hence $(\hat{L}\psi)_{nm}=-\psi_{n,m+1}$, but this is equivalent to $L\psi=0$. The case $n+m\equiv 1 \pmod 2$ is analogous. This completes the proof. □

Any set of non-zero constants g_{nm} defines a 'gauge' transformation of operators of the form (2) such that

$$a'_{nm}=g^{-1}_{n-1,m}a_{nm}\qquad b'_{nm}=g^{-1}_{n+1,m}b_{nm}\qquad c'_{nm}=g^{-1}_{n,m-1}c_{nm}$$
$$d'_{nm}=g^{-1}_{n,m+1}d_{nm}\qquad v'_{nm}=g^{-1}_{nm}v_{nm}.$$

This gauge transform acts on the eigenfunctions in the following manner: $\psi'_{nm}=g_{nm}\psi_{nm}$.

The following theorem is an easy corollary of theorem 1.

Theorem 1′. *For any set of 'spectral data' consisting of: a non-singular curve Γ of genus g; six points $P_i^\pm\in\Gamma$, $i=1,2,3$; and a generic divisor $\mathcal{D}$ of g points different from the $P_i^\pm$, there exists, up to a gauge transformation, a unique operator L of the form (2).*

4. The hexagonal operators: Novikov's class

Consider a triangular lattice in a plane. We will use as coordinates triples of integers k, l, m such that $k + l + m = 0$.

Consider an arbitrary two-dimensional difference operator L of the form (3). Our goal is to find some solution of the generalized inverse problem stated in the introduction.

Our construction is as follows. Let Γ be a non-singular curve of genus g. Let Q_i, R_i, $i = 1, 2, 3$, be six points on Γ. Let $\mathcal{D}$ be a generic divisor of the form $\mathcal{D} = P_1 + \cdots + P_g$ such that the points P_k are different from the Q_i, R_i. Consider a function $\phi_{\alpha\beta\gamma\rho\sigma\tau}$, α, β, γ, ρ, σ, $\tau \in \mathbb{Z}$, $\alpha + \beta + \gamma = 0$, $\rho + \sigma + \tau = 0$, defined on Γ such that:

(1) If a point $P \in \Gamma \setminus \{Q_1, Q_2, Q_3, R_1, R_2, R_3\}$ is a pole of $\phi_{\alpha\beta\gamma\rho\sigma\tau}$, then P is one of the points P_k.
(2) The function $\phi_{\alpha\beta\gamma\rho\sigma\tau}$ has a pole of αth order in Q_1, a pole of βth order in Q_2, a pole of γth order in Q_3; the same structure for ρ, σ, τ and R_1, R_2, R_3.

Lemma.

(1) Such a function $\phi_{\alpha\beta\gamma\rho\sigma\tau}$ exists and is unique up to multiplication by a constant.
(2) The explicit formula for $\phi_{\alpha\beta\gamma\rho\sigma\tau}$ is

$$r_{\alpha\beta\gamma\rho\sigma\tau} \cdot \exp \int_{P_0}^{P} (\alpha\Omega_{Q_3Q_1} + \beta\Omega_{Q_3Q_2} + \rho\Omega_{R_3R_1} + \sigma\Omega_{R_3R_2}) \times \frac{\Theta(A(P) + \alpha U_{Q_3Q_1} + \beta U_{Q_3Q_2} + \rho U_{R_3R_1} + \sigma U_{R_3R_2} - A(\mathcal{D}) - \mathcal{K})}{\Theta(A(P) - A(\mathcal{D}) - \mathcal{K})}$$

where $r_{\alpha\beta\gamma\rho\sigma\tau}$ is an arbitrary constant, P_0 is a fixed point defining the Abel transform A (it should be noted that the paths of integration in $\int_{P_0}^{P}$ and in the Abel transform are the same).

Proof. The proof is performed by standard reasoning of the theory of the algebro-geometric integration. □

As in section 3 we will use vectorial notation. We will represent the six integer numbers $\alpha, \beta, \gamma, \rho, \sigma, \tau$ as one vector

$$v = \alpha e_1 + \beta e_2 + \gamma e_3 + \rho e_4 + \sigma e_5 + \tau e_6 \in \mathbb{Z}^6.$$

Thus, we will write ϕ_v instead of $\phi_{\alpha\beta\gamma\rho\sigma\tau}$.

The key idea of the construction of our functions ψ_{klm} is a convenient relabelling with $\psi_{klm} = \phi_{v(k,l,m)}$, where

$$v(k,l,m) = \frac{k-l}{3}e_1 + \frac{l-m}{3}e_2 + \frac{m-k}{3}e_3 + \frac{k-l}{3}e_4 + \frac{l-m}{3}e_5 + \frac{m-k}{3}e_6 \qquad \text{if} \quad k-l = 0 \pmod 3$$

$$v(k,l,m) = \frac{k-l-1}{3}e_1 + \frac{l-m+2}{3}e_2 + \frac{m-k-1}{3}e_3 + \frac{k-l+2}{3}e_4 + \frac{l-m-1}{3}e_5 + \frac{m-k-1}{3}e_6 \qquad \text{if} \quad k-l = 1 \pmod 3$$

$$v(k,l,m) = \frac{k-l+1}{3}e_1 + \frac{l-m+1}{3}e_2 + \frac{m-k-2}{3}e_3 + \frac{k-l+1}{3}e_4 + \frac{l-m-2}{3}e_5 + \frac{m-k+1}{3}e_6 \qquad \text{if} \quad k-l = 2 \pmod 3.$$

We will also use the following notation:

$$\Theta(P, \alpha e_1 + \beta e_2 + \gamma e_3 + \rho e_4 + \sigma e_5 + \tau e_6) = \Theta(P, \alpha, \beta, \gamma, \rho, \sigma, \tau)$$
$$= \Theta(A(P) + \alpha U_{Q_3 Q_1} + \beta U_{Q_3 Q_2} + \rho U_{R_3 R_1} + \sigma U_{R_3 R_2} - A(\mathcal{D}) - \mathcal{K}).$$

Let us formulate our theorem.

Theorem 2. *Let a family ψ_{klm} be defined as stated above. Then $L\psi = 0$ if and only if the coefficients a_{klm}, b_{klm}, c_{klm}, d_{klm}, f_{klm}, g_{klm}, of the operator L are defined up to a multiplication by a constant by the following formulae:*

(1) if $k - l \equiv 0 \pmod 3$, then

$$\frac{a_{klm}}{b_{klm}} = \frac{r_{v+e_4-e_5}}{r_{v+e_2-e_3}} \cdot \left[\frac{\Theta(Q_2, v - e_1 + e_2)\Theta(Q_3, v + e_4 - e_5)}{\Theta(Q_2, v + e_2 - e_3)\Theta(Q_3, v - e_1 + e_2)} \times \exp\left(\int_{P_0}^{Q_3} (\Omega_{R_2R_1} - \Omega_{Q_1Q_2}) - \int_{P_0}^{Q_2} \Omega_{Q_3Q_1} \right) + \frac{\Theta(Q_2, v + e_2 - e_3 + e_4 - e_6)\Theta(R_1, v + e_4 - e_5)}{\Theta(Q_2, v + e_2 - e_3)\Theta(R_1, v + e_2 - e_3 + e_4 - e_6)} \times \exp\left(\int_{P_0}^{Q_2} \Omega_{R_3R_1} - \int_{P_0}^{R_1} (\Omega_{Q_3Q_2} + \Omega_{R_3R_2}) \right) \right]$$

$$\frac{d_{klm}}{b_{klm}} = -\frac{r_{v+e_4-e_5}}{r_{v-e_1+e_2}} \cdot \frac{\Theta(Q_3, v + e_4 - e_5)}{\Theta(Q_3, v - e_1 + e_2)} \cdot \exp\left(\int_{P_0}^{Q_3} (\Omega_{R_2R_1} - \Omega_{Q_1Q_2}) \right)$$

$$\frac{f_{klm}}{b_{klm}} = -\frac{r_{v+e_4-e_5}}{r_{v+e_2-e_3+e_4-e_6}} \cdot \frac{\Theta(R_1, v + e_4 - e_5)}{\Theta(Q_3, v + e_2 - e_3 + e_4 - e_6)} \times \exp\left(\int_{P_0}^{R_1} (-\Omega_{Q_3Q_2} - \Omega_{R_3R_2}) \right)$$

$$\frac{c_{klm}}{b_{klm}} = 0 \qquad \frac{g_{klm}}{b_{klm}} = 0 \qquad \text{where } v = v(k, l, m).$$

(2) if $k - l \equiv 1 \pmod 3$, then

$$\frac{b_{klm}}{d_{klm}} = -\frac{r_{v-e_4+e_6}}{r_{v+e_1-e_2-e_5+e_6}} \cdot \frac{\Theta(R_3, v - e_4 + e_6)}{\Theta(R_3, v + e_1 - e_2 - e_5 + e_6)} \times \exp\left(\int_{P_0}^{R_3} (\Omega_{Q_1Q_2} + \Omega_{R_1R_2}) \right)$$

$$\frac{c_{klm}}{d_{klm}} = \frac{r_{v-e_4+e_6}}{r_{v+e_1-e_2}} \cdot \left[\frac{\Theta(R_1, v + e_1 - e_2 - e_5 + e_6)\Theta(R_3, v - e_4 + e_6)}{\Theta(R_1, v + e_1 - e_2)\Theta(R_3, v + e_1 - e_2 - e_5 + e_6)} \times \exp\left(\int_{P_0}^{R_3} (\Omega_{Q_1Q_2} + \Omega_{R_1R_2}) - \int_{P_0}^{R_1} \Omega_{R_3R_2} \right) + \frac{\Theta(R_1, v + e_1 - e_3)\Theta(Q_2, v - e_4 + e_6)}{\Theta(R_1, v + e_1 - e_2)\Theta(Q_2, v + e_1 - e_3)} \times \exp\left(\int_{P_0}^{R_1} \Omega_{Q_3Q_2} - \int_{P_0}^{Q_2} (\Omega_{Q_3Q_1} + \Omega_{R_3R_1}) \right) \right]$$

$$\frac{f_{klm}}{d_{klm}} = -\frac{r_{v-e_4+e_6}}{r_{v+e_1-e_3}} \cdot \frac{\Theta(Q_2, v - e_4 + e_6)}{\Theta(Q_2, v + e_1 - e_3)} \cdot \exp\left(\int_{P_0}^{Q_2} (-\Omega_{Q_3Q_1} - \Omega_{R_3R_1}) \right)$$

$$\frac{a_{klm}}{d_{klm}} = 0 \qquad \frac{g_{klm}}{d_{klm}} = 0 \qquad \text{where } v = v(k, l, m).$$

(3) if $k - l \equiv 2 \pmod 3$*, then*

$$\frac{b_{klm}}{f_{klm}} = -\frac{r_{v+e_5-e_6}}{r_{v-e_2+e_3}} \cdot \frac{\Theta(Q_1, v + e_5 - e_6)}{\Theta(Q_1, v - e_2 + e_3)} \cdot \exp\left(\int_{P_0}^{Q_1} (\Omega_{Q_3Q_2} + \Omega_{R_3R_2})\right)$$

$$\frac{d_{klm}}{f_{klm}} = -\frac{r_{v+e_5-e_6}}{r_{v-e_1+e_3-e_4+e_5}} \cdot \frac{\Theta(R_2, v + e_5 - e_6)}{\Theta(R_2, v - e_1 + e_3 - e_4 + e_5)} \times \exp\left(\int_{P_0}^{R_2} (\Omega_{Q_3Q_1} + \Omega_{R_3R_1})\right)$$

$$\frac{g_{klm}}{f_{klm}} = \frac{r_{v+e_5-e_6}}{r_{v-e_1+e_3}} \cdot \left[\frac{\Theta(Q_3, v - e_2 + e_3)\Theta(Q_1, v + e_5 - e_6)}{\Theta(Q_3, v - e_1 + e_3)\Theta(Q_1, v - e_2 + e_3)} \times \exp\left(\int_{P_0}^{Q_1} (\Omega_{Q_3Q_2} + \Omega_{R_3R_2}) - \int_{P_0}^{Q_3} \Omega_{Q_1Q_2}\right) + \frac{\Theta(Q_3, v - e_1 + e_3 - e_4 + e_5)\Theta(R_2, v + e_5 - e_6)}{\Theta(Q_3, v - e_1 + e_3)\Theta(R_2, v - e_1 + e_3 - e_4 + e_5)} \times \exp\left(\int_{P_0}^{R_2} (\Omega_{Q_3Q_1} + \Omega_{R_3R_1}) + \int_{P_0}^{Q_3} \Omega_{R_1R_2}\right)\right]$$

$$\frac{a_{klm}}{f_{klm}} = 0 \qquad \frac{c_{klm}}{f_{klm}} = 0 \qquad \textit{where } v = v(k, l, m).$$

Proof. The proof is analogous to the proof of theorem 1. □

Any set of non-zero constants h_{klm} defines a 'gauge' transformation of operators of the form (3) such that

$$a'_{klm} = h^{-1}_{k,l+1,m-1} a_{klm} \qquad b'_{klm} = h^{-1}_{k,l-1,m+1} b_{klm} \qquad c'_{klm} = h^{-1}_{k+1,l-1,m} c_{klm}$$
$$d'_{klm} = h^{-1}_{k-1,l+1,m} d_{klm} \qquad f'_{klm} = h^{-1}_{k+1,l,m-1} f_{klm} \qquad g'_{klm} = h^{-1}_{k-1,l,m+1} g_{klm}.$$

This gauge transform acts on the eigenfunctions in the following manner: $\psi'_{klm} = h_{klm}\psi_{klm}$.

The following theorem is an easy corollary of theorem 2.

Theorem 2′. *For any set of 'spectral data' consisting of: a non-singular curve* Γ *of genus* g*, six points* $Q_i, R_i \in \Gamma, i = 1, 2, 3$*, and a generic divisor* $\mathcal{D}$ *of* g *points different from the* Q_i, R_i*, there exists, up to a gauge transformation, a unique operator* L *of the form (3).*

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